Ultrafilters: an algebraic description of topological dynamics

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Torino - 10/03/2022
Algebra and topology of ultrafilters on $\mathbb{N}$

Topological dynamics and ultrafilters

Partition regular properties and shift dynamics
Section 1

Algebra and topology of ultrafilters on $\mathbb{N}$
Filters and ultrafilters

Definition

A **filter** on \( \mathbb{N} \) is a nonempty family \( p \subseteq \mathcal{P}(\mathbb{N}) \) such that:

- if \( A, B \in p \) then \( A \cap B \in p \);
- if \( A \subseteq B, A \in p \) then \( B \in p \);
- \( \emptyset \notin p \).

A **ultrafilter** is a maximal filter. Equivalently, it is a filter \( p \) such that if \( A \notin p \) then \( A^c \in p \).

For all \( n \in \mathbb{N} \), \( \mathcal{U}_n := \{ A \subseteq \mathbb{N} : n \in A \} \) is an ultrafilter, the **principal ultrafilter generated by** \( n \).

An ultrafilter is non-principal if and only if it extends the **Frechet filter** \( \{ A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ finite} \} \).
The space of ultrafilters

Let $\beta \mathbb{N}$ be the set of ultrafilters on $\mathbb{N}$.

For every $B \subseteq \mathbb{N}$ let

$$N_B := \{ p \in \beta \mathbb{N} : B \in p \}.$$

Then $\{ N_B : B \subseteq \mathbb{N} \}$ is a clopen base for a compact Hausdorff topology on $\beta \mathbb{N}$.

Identify $\mathbb{N}$ with the set $\{ U_n : n \in \mathbb{N} \}$ of principal ultrafilters. For $B \subseteq \mathbb{N}$

$$\overline{B} = \overline{\{ U_n : n \in B \}} = N_B.$$
Ultrafilter limits

Let $X$ be a topological space and let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in $X$. Let $p \in \beta \mathbb{N}$. The $p$-limit $p - \lim_n x_n$ is a point $y \in X$ such that for every $U \subseteq X$ open neighborhood of $y$,

$$\{ n \in \mathbb{N} : x_n \in U \} \in p.$$

Proposition

- If $X$ is compact, there exists at least one $p$-limit for every $p \in \beta \mathbb{N}$;
- if $X$ is Hausdorff, there exists at most one $p$-limit for every $p \in \beta \mathbb{N}$.

Theorem (Stone-Cech compactification of $\mathbb{N}$)

For every $f : \mathbb{N} \to K$ with $K$ compact Hausdorff there exists a unique continuous map $\beta f : \beta \mathbb{N} \to K$ such that $\beta f(n) = f(n)$ for every $n \in \mathbb{N}$. 

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Sketch.

Define $\beta f : \beta \mathbb{N} \to K$ by letting

$$\beta f(p) := p - \lim_{n} f(n).$$

$\beta f$ is well defined since $K$ compact Hausdorff.

Corollary

Let $f : X \to Y$ be a continuous map between compact Hausdorff spaces. For every $\{x_n\}_{n \in \mathbb{N}} \subset X$ and for every $p \in \beta \mathbb{N}$

$$f(p - \lim_{n} x_n) = p - \lim_{n} f(x_n).$$
The semigroup of ultrafilters

We want to extend the addition of $\mathbb{N}$ to $\beta\mathbb{N}$. If $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$,

$$A - n = \{ m \in \mathbb{N} : n + m \in A \}.$$ 

If $p \in \beta\mathbb{N}$,

$$A - p := \{ n \in \mathbb{N} : A - n \in p \}.$$ 

Define an associative operation $\oplus$ on $\beta\mathbb{N}$ by letting

$$A \in p \oplus q \iff A - q \in p.$$ 

For all $n, m \in \mathbb{N}$, $\mathcal{U}_n \oplus \mathcal{U}_m = \mathcal{U}_{n+m}$. 

For all $p, q \in \beta\mathbb{N}$, $p \oplus q = p - \lim_n (q - \lim_m (n + m))$. 

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The addition \( \oplus \) in \( \beta \mathbb{N} \) is not commutative: if \( p \oplus q = q \oplus p \) for every \( q \in \beta \mathbb{N} \), then \( p = \mathcal{U}_n \) for some \( n \in \mathbb{N} \).

Proposition

\((\beta \mathbb{N}, \oplus)\) is a right topological semigroup, i.e. the maps

\[
\lambda_q : \beta \mathbb{N} \to \beta \mathbb{N} \\
\quad p \mapsto p \oplus q
\]

are continuous for every \( q \in \beta \mathbb{N} \).

On the other hand, the map \( q \mapsto p \oplus q \) is continuous if and only if \( p = \mathcal{U}_n \) for some \( n \in \mathbb{N} \).

Theorem (Ellis’ Lemma)

In a compact Hausdorff right topological semigroup \((S, \ast)\) there are idempotents, i.e. \( s \in S \) such that \( s \ast s = s \).
Ideals and minimal ideals

Definition
A subset $L \subseteq \beta\mathbb{N}$ is a left ideal if $\beta\mathbb{N} \oplus L \subseteq L$, i.e. $p \oplus q \in L$ for every $p \in \beta\mathbb{N}$ and $q \in L$.

$R \subseteq \beta\mathbb{N}$ is a right ideal if $R \oplus \beta\mathbb{N} \subseteq R$.

A two sided ideal is a left ideal which is also a right ideal.

Proposition
In $\beta\mathbb{N}$, every left (right) ideal contains a minimal left (right) ideal.
Moreover, in $\beta\mathbb{N}$ there is a unique minimal two sided ideal, called $K(\beta\mathbb{N})$.
$K(\beta\mathbb{N})$ is the union of all the minimal left ideals and it is also the union of all the minimal right ideals.
Section 2

Topological dynamics and ultrafilters
A topological dynamics is a pair \((X, T)\) where \(X\) is a compact Hausdorff space and \(T : X \to X\) is a continuous map. Define \(T^1 := T\) and \(T^{n+1} := T \circ T^n\).

If \(x \in X\), the orbit of \(x\) is \(\text{orb}(x) := \{T^n x : n \in \mathbb{N}\}\).

The topological dynamics generated by \(x\) is the topological dynamics \((X(x), T)\) where \(X(x) = \text{orb}(x)\).

For \(p \in \beta\mathbb{N}\), define \(T^p x := p - \lim_n T^n x\). Then
\[
X(x) = \{T^p x : p \in \beta\mathbb{N}\}.
\]
Consider now the case \((X, T) = (\beta\mathbb{N}, S)\), where \(S(q) = q \oplus 1\).

If \(q \in \beta\mathbb{N}\), \(S^p q = p \oplus q\) and \(\beta\mathbb{N}(q)\) is the left ideal generated by \(q\)

\[
\beta\mathbb{N}(q) = \beta\mathbb{N} \oplus q = \{p \oplus q : p \in \beta\mathbb{N}\} = \lambda_q[\beta\mathbb{N}].
\]

For every \((X, T)\) topological dynamics and for every \(x \in X\), we have a commutative diagram

\[
\begin{array}{ccc}
\beta\mathbb{N} & \xrightarrow{S} & \beta\mathbb{N} \\
\tau_x \downarrow & & \tau_x \downarrow \\
X(x) & \xrightarrow{T} & X(x)
\end{array}
\]

where \(\tau_x : p \mapsto T^p x\) is continuous and surjective. Moreover,

\[
T^p T^q x = T^{p \oplus q} x.
\]
Recurrence

Definition
Let \((X, T)\) a topological dynamics. A point \(x \in X\) is recurrent if for every \(U\) open neighborhood of \(x\) the set \(R_x(U) := \{ n \in \mathbb{N} : T^n x \in U \}\) is nonempty.

Proposition
For \(x \in X\) the following are equivalent:

1. \(x\) is recurrent;
2. there exists a non-principal \(p \in \beta \mathbb{N}\) such that \(T^p x = x\);
3. there exists an idempotent \(p \in \beta \mathbb{N}\) such that \(T^p x = x\).
In $\beta \mathbb{N}$ recurrent ultrafilters are exactly of the form $p \oplus q$ with $p$ idempotent:

Corollary

$$\{ q \in \beta \mathbb{N} : q \text{ is recurrent} \} = \bigcup \{ p \oplus \beta \mathbb{N} : p = p \oplus p \}.$$ 

Corollary

Let $(X, T)$ be a topological dynamics and let $x \in X$. Then $y \in X(x)$ is recurrent if and only if there exists $q \in \beta \mathbb{N}$ recurrent such that $T^q x = y.$
Uniform recurrence

Definition
Let \((X, T)\) a topological dynamics. A point \(x \in X\) is \textit{uniformly recurrent} if for every \(U\) open neighborhood of \(x\) the set \(R_x(U) := \{n \in \mathbb{N} : T^n x \in U\}\) is \textit{syndetic}, i.e. there exists \(k\) such that for every \(n \in \mathbb{N}\) at least one among \(n, n + 1, \ldots, n + k\) is in \(R_x(U)\).

Proposition
For \(x \in X\) the following are equivalent:

1. \(x\) is uniformly recurrent;
2. \((X(x), T)\) is a minimal dynamical system;
3. for every \(y \in X(x), X(y) = X(x)\);
4. for every \(q \in \beta\mathbb{N}\) there exists \(p \in \beta\mathbb{N}\) such that \(T^{p \oplus q}x = x\);
5. there exists an idempotent \(p \in K(\beta\mathbb{N})\) such that \(T^p x = x\).
In $\beta\mathbb{N}$, the minimal dynamical subsystems are exactly the minimal left ideals.

**Corollary**

$$\{q \in \beta\mathbb{N} : q \text{ uniformly recurrent} \} = K(\beta\mathbb{N}) = \bigcup \{p \oplus \beta\mathbb{N} : p = p \oplus p \in K(\beta\mathbb{N})\}.$$

**Corollary**

Let $(X, T)$ be a topological dynamics and let $x \in X$. Then $y \in X(x)$ is uniformly recurrent if and only if there exists $q \in K(\beta\mathbb{N})$ such that $T^q x = y$. 
Proximality

Definition
Let \((X, T)\) be a topological dynamics. Call \(\Delta_X : = \{(x, x) : x \in X\}\) the diagonal. Two points \(x, y \in X\) are **proximal** if for every open neighborhood \(U\) of \(\Delta_X\) there are infinitely many \(n \in \mathbb{N}\) such that \((T^n x, T^n y) \in U\).

Proposition

For \(x, y \in X\) the following are equivalent:

1. \(x\) and \(y\) are proximal;
2. for every finite open cover \(X = U_1 \cup \cdots \cup U_r\), there exist \(i\) and \(n\) such that \(T^n x, T^n y \in U_i\);
3. there exists a nonprincipal \(p \in \beta\mathbb{N}\) such that \(T^p x = T^p y\);
4. there exists a minimal idempotent \(p \in \beta\mathbb{N}\) such that \(T^p x = T^p y\).
Section 3

Partition regular properties and shift dynamics
**Definition**

Let $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ be such that if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$. \(\mathcal{F}\) is *(strongly) partition regular* if for every $F \in \mathcal{F}$ and for every finite partition $F = C_1 \cup \cdots \cup C_r$ at least one $C_i$ is in $\mathcal{F}$.

**Proposition**

$\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ is strongly partition regular if and only if there exists a family $\{p_i : i \in I\} \subseteq \beta\mathbb{N}$ such that $\bigcup_{i \in I} p_i = \mathcal{F}$. 
Hindman’s Theorem

Definition

$A \subseteq \mathbb{N}$ is an **IP set** if there exists a sequence $S = \{s_i\}_{i \in \mathbb{N}} \subseteq \mathbb{N}$ such that

$$A \supseteq \text{FS}(S) := \{s_{i_1} + \cdots + s_{i_k} : k, i_1, \ldots, i_k \in \mathbb{N}\}.$$

Theorem

$$\{A \subseteq \mathbb{N} : A \text{ IP set}\} = \bigcup \{p \in \beta\mathbb{N} : p = p \oplus p\}.$$

Corollary (Hindman’s Theorem)

*For every finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, at least one $C_i$ is an IP set.*

Corollary

*Let $(X, T)$ be a topological dynamics and let $x \in X$ be recurrent. Then for every $U$ open neighborhood of $x$ $R_x(U) = \{n \in \mathbb{N} : T^n x \in U\}$ is an IP set.*
Identify $\mathcal{P}(\mathbb{N})$ with the Cantor space $2^{\mathbb{N}}$. Its topology is compact metric zero-dimensional. Basic clopen sets are

$$\Lambda^E_F = \{ B \in \mathcal{P}(\mathbb{N}) : E \subseteq B, B \cap F = \emptyset \}, \text{ for } E, F \subseteq \mathbb{N} \text{ finite}.$$ 

Take the continuous map

$$s : B \mapsto B - 1 := \{ m \in \mathbb{N} : m + 1 \in B \}.$$ 

Then $(\mathcal{P}(\mathbb{N}), s)$ is a topological dynamics. The unique fixed points of $s$ are $\emptyset$ and $\mathbb{N}$. Moreover,

$$s^n : B \mapsto B - n, \quad s^p : B \mapsto B - p = \{ n \in \mathbb{N} : B - n \in p \}.$$ 

In particular, $\mathcal{P}(\mathbb{N})(A) = \{ A - p : p \in \beta\mathbb{N} \}$. 

Recurrence in shift dynamics

Corollary

Recurrent points in the closure of the orbit of $A \in \mathcal{P}(\mathbb{N})$ are all the $A - p$ with $p$ recurrent. Moreover, $A$ is recurrent if and only if $A - p = A$ for some $p = p \oplus p$.

In particular, $A$ is recurrent if and only if $A^c$ is recurrent.

Theorem

For $p \in \beta \mathbb{N}$ the following are equivalent:

1. $p$ is recurrent;
2. for every $A \in p$, $A - p$ is an IP set;
3. for every $A \in \mathcal{P}(\mathbb{N})$, $A - p$ is recurrent.
Theorem

For $A \in \mathcal{P}(\mathbb{N})$ the following are equivalent:

1. there exists a recurrent $p \in \beta\mathbb{N}$ such that $A \in p$;
2. $A - q$ is an IP set for some $q \in \beta\mathbb{N}$;
3. in the closure of the orbit of $A$ there is a nonempty recurrent set;
4. in the closure of the orbit of $A$ there is a recurrent set which is an IP set.
**Piecewise syndeticity**

**Definition**

Let $A \subseteq \mathbb{N}$. Then

- $A$ is **syndetic** if there exists $k \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ at least one among $n + 1, \ldots, n + k$ is in $A$;
- $A$ is **thick** if $A^c$ is not syndetic;
- $A$ is **piecewise syndetic** if $A = S \cap T$ with $S$ syndetic and $T$ thick.

*The families of syndetic sets and thick sets are not partition regular.*

**Proposition**

$$\{A \in \mathcal{P}(\mathbb{N}) : A \text{ is piecewise syndetic}\} = \bigcup K(\beta \mathbb{N}).$$

In particular, being piecewise syndetic is partition regular.
Corollary

Uniformly recurrent points in the closure of the orbit of $A \in \mathcal{P}(\mathbb{N})$ are all the $A - p$ with $p \in K(\beta\mathbb{N})$.
Moreover, $A$ is uniformly recurrent if and only if $A - p = A$ for some $p = p \oplus p \in K(\beta\mathbb{N})$.
In particular, $A$ is uniformly recurrent if and only if $A^c$ is uniformly recurrent.

Theorem

For $p \in \beta\mathbb{N}$ the following are equivalent:

1. $p \in K(\beta\mathbb{N})$;
2. for every $A \in p$, $A - p$ is an syndetic;
3. for every $A \in \mathcal{P}(\mathbb{N})$, $A - p$ is uniformly recurrent.
Theorem

For $A \in \mathcal{P}(\mathbb{N})$ the following are equivalent:

1. there exists $p \in K(\beta\mathbb{N})$ such that $A \in p$;
2. $A - q$ is an syndetic for some $q \in \beta\mathbb{N}$;
3. in the closure of the orbit of $A$ there is a nonempty uniformly recurrent set;
4. in the closure of the orbit of $A$ there is a syndetic uniformly recurrent set.
Let $A \subseteq \mathbb{N}$. The uniformly recurrent sets in the closure of the orbit of $A$ depend on the shape of the set $A$:

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<th>$\mathbb{N}$</th>
<th>other</th>
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<td>$x$</td>
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<tr>
<td>$A$ PS not syndetic, not thick</td>
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<td>$A$ thick syndetic, $A^c$ not PS</td>
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References


THANK YOU!