Classical descriptive set theory, generalized descriptive set theory, and I0

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G. Cantor (1870) proved that all closed subsets of $\mathbb{R}$ have the PSP: this is one of the earlier results in the area now called (classical) descriptive set theory.

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The resulting theory is extremely rich and interesting, but quite different from the classical one: most of the nontrivial results are either simply false or at least independent of ZFC when $\kappa > \omega$ (e.g. both the *Lusin's separation theorem* and *Souslin's theorem* fail).

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More recently, Woodin suggested to study generalized DST under I0 in connection with his study of the model $\mathrm{L}(\mathrm{V}_{\lambda+1})$ (where λ is the witness of I0). Notice that such a λ has always countable cofinality.

I0 and Woodin's analysis

The axiom I0

I0(λ) is the statement: There is a nontrivial elementary embedding $j \colon \mathrm{L}(\mathrm{V}_{\lambda+1}) \prec \mathrm{L}(\mathrm{V}_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ (we call j a *witness* to I0(λ)).

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$$O_{a,\alpha} = \{X \in V_{\lambda+1} \mid X \cap \mathrm{V}_\alpha = a\}$$

for $\alpha < \lambda$ and $a \subseteq \mathrm{V}_\alpha$.

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Woodin claims that "*the theory of* $\mathcal{P}(\mathrm{V}_{\lambda+1})$ *in* $\mathrm{L}(\mathrm{V}_{\lambda+1})$ *under* I0(λ) *is reminiscent of the theory of* $\mathcal{P}(\mathbb{R})$ *in* $\mathrm{L}(\mathbb{R}) = \mathrm{L}(\mathrm{V}_{\omega+1})$ *under* AD".

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Theorem (Woodin)

Assume I0(λ), as witnessed by j. Every $U(j)$-representable set $A \subseteq \mathrm{V}_{\lambda+1}$ in $\mathrm{L}(\mathrm{V}_{\lambda+1})$ satisfies the following dichotomy: either $|A| \leq \lambda$ or ${}^{\omega}2$ topologically embeds into A.

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Assume I0(λ), as witnessed by j. Then every set A in $\mathrm{L}_\lambda(\mathrm{V}_{\lambda+1})$ satisfies the following dichotomy: either $|A| \leq \lambda$ or $C(\lambda) = \prod_{i \in \omega} \lambda_i$ topologically embeds into A, where $\lambda_i \nearrow \lambda$.

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Except for Woodin's result (which is the weakest!), the proofs are quite different from the classical ones dealing with the (classical) PSP, and indeed technical machineries specific to the model $\mathrm{L}(\mathrm{V}_{\lambda+1})$ under I0(λ) are heavily involved.

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Our goal is to study the generalized Cantor space ${}^{\lambda}2$ when λ is singular.
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- ${}^{\mu}(2^{<\lambda})$, where $2^{<\lambda}$ is discrete.

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Dropping the first half of the usual condition

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A metric space X is said **uniformly zero-dimensional** if for every $\varepsilon > 0$, every open set of X can be partitioned into *clopen* sets with diameter $< \varepsilon$.

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  (A space X is a uniformly zero-dimensional λ-Polish space iff it is homeomorphic to a closed subset of ${}^{\lambda}2$, iff it admits a compatible complete ultrametric).

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- Every nonempty λ-Polish space is a continuous image of ${}^{\lambda}2$.

The generalized Cantor space ${}^{\lambda}2$ and Woodin's $\mathrm{L}(\mathrm{V}_{\lambda+1})$

Woodin's approach to the study of $\mathrm{V}_{\lambda+1}$ falls in this setup as well. Recall that $\mathrm{V}_{\lambda+1}$ is endowed with the topology generated by $O_{a,\alpha} = \{X \in V_{\lambda+1} \mid X \cap \mathrm{V}_\alpha = a\}$ for $\alpha < \lambda$ and $a \subseteq \mathrm{V}_\alpha$.

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If $\mathrm{cof}(\lambda) = \omega$ and $\lambda_i \nearrow \lambda$, then

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If furthermore λ is limit of inaccessible cardinals (which is the case under I0(λ)), then

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Similar results hold for the generalized Baire space ${}^{\lambda}\lambda$.

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There are some problems when trying to generalize these equivalences by replacing ${}^{\omega}2$ and ${}^{\omega}\omega$ with ${}^{\kappa}2$ and ${}^{\kappa}\kappa$, especially when κ is regular.

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What for more complicated sets?

λ-Perfect set property

Definition

A subset A of a topological space X has the λ-PSP if either $|A| \leq \lambda$, or else ${}^{\lambda}2$ topologically embeds into A.

Similarly to the classical case

Theorem (essentially A. H. Stone)

Let λ be strong limit of countable cofinality. Every λ-analytic subset of a uniformly zero-dimensional λ-Polish space has the λ-PSP.

What for more complicated sets?

Motivated by the fact that, in the classical context, κ-homogeneously Souslin sets have the PSP (and inspired by Woodin's notion of $U(j)$-representability), we developed the following machinery.

$(\mathbb{U}, \kappa)$-representable sets

Definition

A family $\mathbb{U}$ of ultrafilters is **orderly** iff there exists a set K such that for all $\mathcal{U} \in \mathbb{U}$ there is $n \in \omega$ for which ${}^nK \in \mathcal{U}$. Such an n is called the **level** of $\mathcal{U}$.

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A **tower** of ultrafilters in such a $\mathbb{U}$ is a sequence $(\mathcal{U}_i)_{i \in \omega}$ such that for all $m < n < \omega$:

- $\mathcal{U}_n \in \mathbb{U}$ has level n;
- $\mathcal{U}_n$ *projects* to $\mathcal{U}_m$, i.e. for each $A \subseteq {}^mK$ we have

$$A \in \mathcal{U}_m \iff \{s \in {}^nK \mid s \restriction m \in A\} \in \mathcal{U}_n.$$

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A tower of ultrafilters $(\mathcal{U}_i)_{i \in \omega}$ is **well-founded** iff for every sequence $(A_i)_{i \in \omega}$ with $A_i \in \mathcal{U}_i$ there is $z \in {}^\omega K$ such that $z \restriction i \in A_i$ for all $i \in \omega$.

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Let $\kappa \geq \lambda$ be a cardinal, and let $\mathbb{U}$ be an orderly family of κ-*complete* ultrafilters. A $(\mathbb{U}, \kappa)$-**representation** for $Z \subseteq {}^{\omega}\lambda$ is a function $\pi \colon \bigcup_{i \in \omega} {}^{i}\lambda \times {}^{i}\lambda \to \mathbb{U}$ such that:

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Remark 2: Exploiting the natural homeomorphism between $\mathrm{V}_{\lambda+1}$ and ${}^{\omega}\lambda$ the above definition yields Woodin's $U(j)$-representability when $\kappa = \lambda^+$ and $\mathbb{U}$ is a certain family of ultrafilters usually denoted by $\mathbb{U}(j, \kappa, (a_i)_{i \in \omega})$.

Tower condition

The following condition turns out to be very helpful when checking well-foundness of towers of ultrafilters.

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Remark 2: Woodin has an analogous notion of "tower condition" in the context of $U(j)$-representability. Cramer later proved that if I0(λ) holds, then all $U(j)$-representable sets in $\mathcal{P}(\mathrm{V}_{\lambda+1}) \cap \mathrm{L}(\mathrm{V}_{\lambda+1})$ admit in fact a $U(j)$-representation with the tower condition.

The main theorem

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Corollary

Assume I0(λ), as witnessed by j. If $A \in \mathcal{P}(\mathrm{V}_{\lambda+1}) \cap \mathrm{L}(\mathrm{V}_{\lambda+1})$ is $U(j)$-representable, then A has the λ-PSP.

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Assume I0(λ). All λ-projective subsets of any uniformly zero-dimensional λ-Polish space have the λ-PSP.

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- $s^k_i, t^k_i \in {}^{j_k}\mu_k$ for some $\mu_k < \lambda$ and $j_k \in \omega$, with $s^k_i \neq s^k_{i'}$ if $i \neq i'$;

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Proof of the main theorem

Theorem (Dimonte-M.)

Let λ be strong limit with $\mathrm{cof}(\lambda) = \omega$, and let $\kappa \geq \lambda$ be a cardinal. If $Z \subseteq {}^{\omega}\lambda$ admits a $(\mathbb{U}, \kappa)$-representation with the tower condition, then Z has the λ-PSP.

Proof. Let π be a $(\mathbb{U}, \kappa)$-representation for Z with the tower condition, as witnessed by F. Let $G(Z)$ (or rather $G(\pi, F)$) be the game

I	$(s_i^0, t_i^0)_{i<\lambda_0}$	$z_0, (s_i^1, t_i^1)_{i<\lambda_1}$	$z_1, (s_i^2, t_i^2)_{i<\lambda_2}$	$\dots$
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I wins if she can play for infinitely many turns.

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When I wins a run, she has built an element $x = \bigcup_{k \in \omega} s^k_{i_k} \in {}^\omega\lambda$, and a $y = \bigcup_{k \in \omega} t^k_{i_k} \in {}^\omega\lambda$

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where I *does not have to produce the witnesses* z_k, and I wins iff $x = \bigcup_{k \in \omega} s^k_{i_k} \in Z$ with $y = t^k_{i_k}$ witnessing this. *A priori*, such a game is not necessarily determined (the complexity of the payoff depends on the complexity of Z and π), but...

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Claim. If II wins $G^*(Z)$, then $|Z| \leq \lambda$.

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Proof of the main theorem

...any winning strategy τ of II in $G(Z)$ can be converted into a winning strategy τ^* of II in $G^*(Z)$.

The idea is that II simulates a run in $G(Z)$ testing all possible $z_k \in F(\pi(s^k_{i_k}, t^k_{i_k}))$ that I could play. Using λ-completeness of $\pi(s^k_{i_k}, t^k_{i_k})$, for a measure-one set of these possibilities τ will suggest the very same move $\bar{\imath}_k$: then II plays precisely this $\bar{\imath}_k$ in his corresponding turn in $G^*(Z)$.

Claim. If II wins $G^*(Z)$, then $|Z| \leq \lambda$.

Given a position p in the game $G^*(Z)$ consisting of k-many rounds, let A_p be the set of those $s^{k-1}_{i_{k-1}} \sqsubseteq x \in {}^\omega\lambda$ for which whatever I plays in her next turn, the answer by II following τ^* is such that $s^k_{i_k} \not\sqsubseteq x$. Arguing as in the classical case, one gets $|A_p| \leq (\lambda_k)^\omega < \lambda$. Moreover, $Z \subseteq \bigcup_p A_p$ because any $x \in Z \setminus \bigcup_p A_p$ would yield a strategy for I in $G^*(Z)$ defeating τ^*. Finally, a direct computation shows that there are only λ-many possible positions p in $G^*(Z)$, whence $|Z| \leq \left|\bigcup_p A_p\right| \leq \lambda \cdot \lambda = \lambda$. □

The end

Thank you for your attention!